What is a Computer Algebra System (CAS) - Basics of CAS

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In this blog, we will explain what is a Computer Algebra System - also known as CAS.

1 Main principle of CAS

A Computer Algebra System is a program dedicated to performing ‘abstract’ algebra - and arithmetical - computations. The algebra can be ‘elementary’ such as the standard operations

: addition, multiplication etc... or integers or it can be extremely advanced, for example group theory, non commutative algebras or C∗ algebras for instance.

CAS are specialized software and do not fall into the ‘popular’ categories of programs, unless the web browsers, web servers, word processing or development environments.

CAS is generally considered to be used in applied mathematics but some relatively recent developments are making possible the use of CAS in fields considered to belong to pure mathematics.

Many of the readers who have been through secondary education have probably encountered CAS in the electronic calculators as sold by CASIO or Texas Instrument.

2 CAS implemented in electronic calculators

The main CAS available in the calculators are the following:

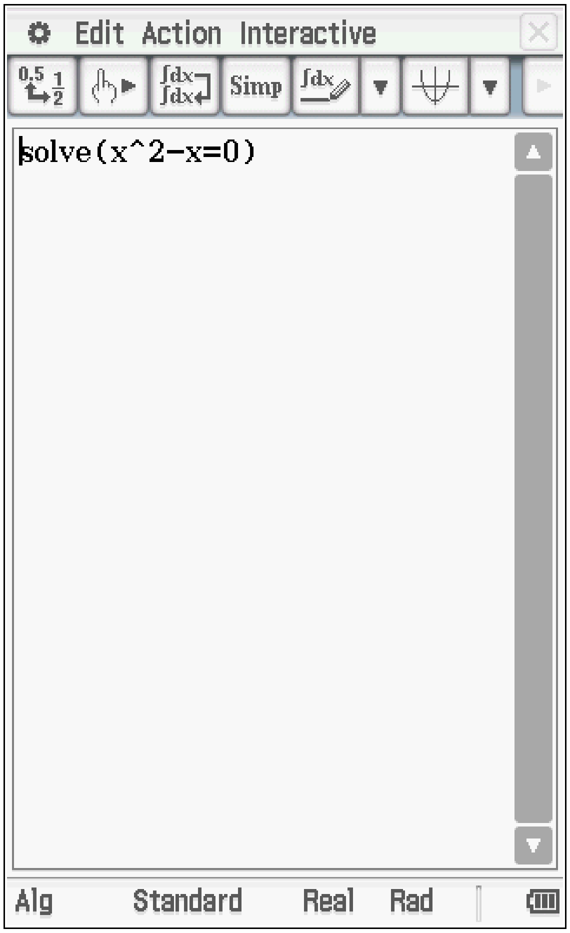
- Derive
- Xcas
- NSpire CAS

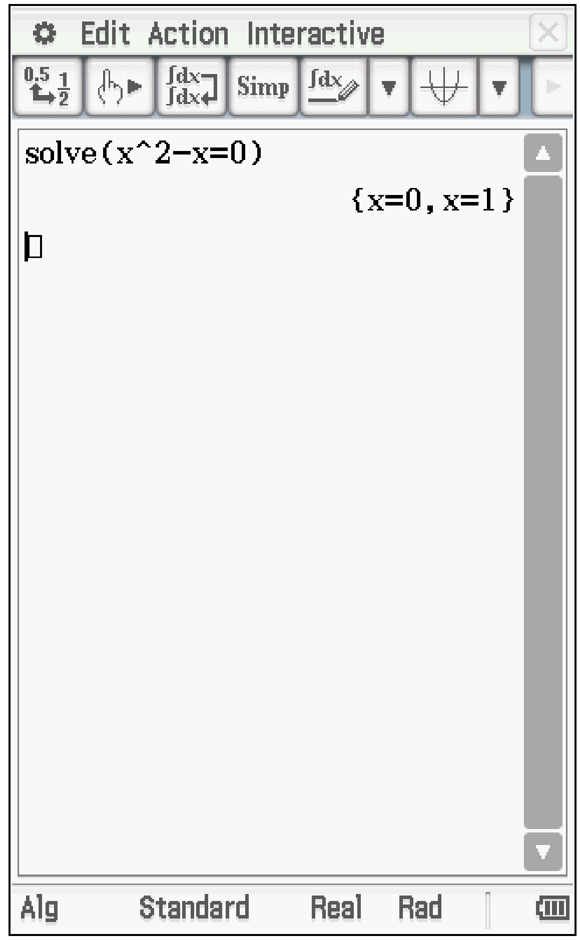
Electronic calculators like the CASIO ClassPad II (fx-CP400) for instance are equipped with such CAS. Here we will look at several features of their Computer Algebra System.

2.1 Solver for algebraic equations of degree 2,3 and 4

We start by testing a small solver. We wish the CAS to find the solutions of:

$x^2 - x = 0$

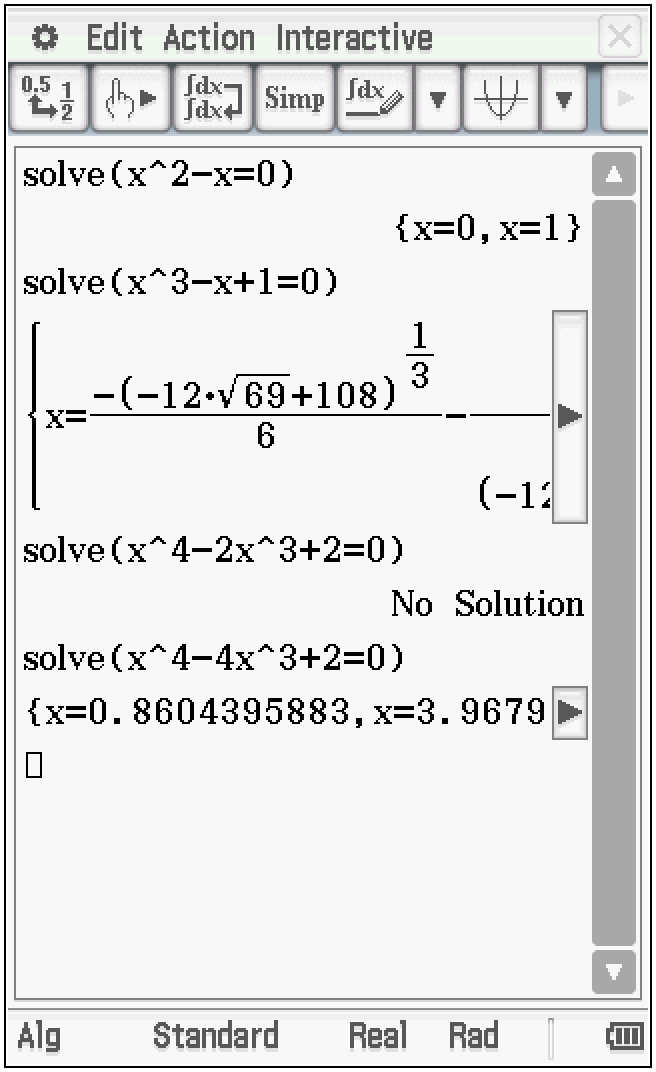




The CAS immediately finds the right solutions, namely x=0 and x=1.

We try a few more difficult equations to solve.

$x^3 - x + 1 = 0$
$x^4 - 2x^3 + 2 = 0$
$x^4 - 4x^3 + 2 = 0$



Here we can see the CAS can solve symbolically polynomial real equations of the third degree.

We check the validity of the solution of the third degree equation:
$x^3 - x + 1 = 0$

The Casio CAS found:

$x = -\frac{\sqrt[3]{108-12\sqrt{69}}}{6} - \frac{2}{\sqrt[3]{108-12\sqrt{69}}}$

We know that the discriminant of a third degree equation$x^3 + px + q - 0$ is

$\Delta = (q/2)^2 + (p/3)^3$

Here $\Delta = (1/2)^2 - (1/3)^3 = 1/4 - 1/27 = 23/108$

Since the discriminant is strictly positive, this means there is one real solution of the equation, and two which are strictly complex.

One of the roots is given by the well-known formula (Cardano's formula)

$$x = \sqrt[3]{-\tfrac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\tfrac{q}{2} - \sqrt{\Delta}}$$

That is to say:

$$x = \sqrt[3]{-\tfrac{1}{2} + \sqrt{23/108}} + \sqrt[3]{-\tfrac{1}{2} - \sqrt{23/108}}$$

Let us check that the CAS found the right result by transforming the algebraic expression:

1. **First term:**

$$\sqrt[3]{-\tfrac{1}{2} + \sqrt{23/108}} = -\tfrac{1}{6}\sqrt[3]{108 - 12\sqrt{69}}$$

1. **Second term:**

$$\sqrt[3]{-\tfrac{1}{2} - \sqrt{23/108}} = \tfrac{1}{6}\sqrt[3]{-108 - 216\sqrt{23/108}} = \tfrac{1}{6}\sqrt[3]{-108 - 12\sqrt{69}}$$
$$= -\frac{1}{6}\frac{\sqrt[3]{108-12\sqrt{69}}\,\sqrt[3]{108+12\sqrt{69}}}{\sqrt[3]{108-12\sqrt{69}}} = -\frac{1}{6}\frac{\sqrt[3]{108^2-12^2.69}}{\sqrt[3]{108-12\sqrt{69}}} = -\frac{1}{6}\frac{\sqrt[3]{11664-9936}}{\sqrt[3]{108-12\sqrt{69}}} = -\frac{2}{\sqrt[3]{108-12\sqrt{69}}}$$

This said, the CAS found *only one* of the roots. The real one. There are two other complex roots.

Let us call x_1 that root, the two other roots x_2, x_3 must satisfy: $S = x_2 + x_3 = -x_1$ and $P = x_2 x_3 = -1/x_1$

Therefore $x_2, x_3 = \{\frac{-S \pm \sqrt{S^2-4P}}{2}\}$

The CAS finds the right formula:

(-S+i√(-S^2+4P))/2

$$\frac{-\frac{(-12\cdot\sqrt{69}+108)^{\frac{1}{3}}}{6}-\frac{2}{(-12\cdot\sqrt{69}+108)^{\frac{1}{3}}} \;+/-\; i\cdot\sqrt{\dfrac{\frac{\sqrt{69}}{3}-(-12\cdot\sqrt{69}+108)^{\frac{1}{3}}-\frac{12}{(-12\cdot\sqrt{69}+108)^{\frac{1}{3}}}-\frac{48}{-12\cdot\sqrt{69}+108}+21}{(-12\cdot\sqrt{69}+108)^{\frac{1}{3}}+\frac{12}{(-12\cdot\sqrt{69}+108)^{\frac{1}{3}}}}}}{2}$$

For the 4th degree equation, the CAS detects when there is no solution but finds only an approach solution instead of using Ferrari's formula.

2.2 Differential operations

The CASIO CAS can perform differential operations

$\frac{d(sin(log(x)-x)}{dx}$.

The result is:

$$\frac{-\cos(\log(x)-x)\cdot(x\cdot\ln(5)+x\cdot\ln(2)-1)}{x\cdot(\ln(5)+\ln(2))}$$

We can compute higher degrees: $\frac{d^2(sin(log(x)-x)}{dx^2}$

The CAS returns:

$$\frac{-\left(x^2\cdot\sin(\log(x)-x)\cdot(\ln(5))^2+x^2\cdot\sin(\log(x)-x)\cdot(\ln(2))^2+2\cdot x^2\cdot\sin(\log(x)-x)\cdot\ln(5)\cdot\ln(2)\right.}{x^2\cdot(\ln(5)+\ln(2))^2}$$

$$\frac{-2\cdot x\cdot\sin(\log(x)-x)\cdot\ln(5)-2\cdot x\cdot\sin(\log(x)-x)\cdot\ln(2)\cdot}{x^2\cdot(\ln(5)+\ln(2))^2}$$

$$\frac{\left.+\cos(\log(x)-x)\cdot\ln(5)+\cos(\log(x)-x)\cdot\ln(2)+\sin(\log(x)-x)\right)}{x^2\cdot(\ln(5)+\ln(2))^2}$$

From the rank 4, the CAS will use approximation to compute the symbolic value of the derivative.

2.3 Factorization and Expansion

The CAS can expand symbolic algebraic expressions:

$$\text{expand}\left(\prod_{i=0}^{10}(x\hat{\ }(-i)+1)\right)$$

$$\frac{2}{x}+\frac{2}{x^2}+\frac{4}{x^3}+\frac{4}{x^4}+\frac{6}{x^5}+\frac{8}{x^6}+\frac{10}{x^7}+\frac{12}{x^8}+\frac{16}{x^9}+\frac{20}{x^{10}}+\frac{22}{x^{11}}+$$

$$+\frac{26}{x^{12}}+\frac{30}{x^{13}}+\frac{34}{x^{14}}+\frac{40}{x^{15}}+\frac{44}{x^{16}}+\frac{48}{x^{17}}+\frac{54}{x^{18}}+\frac{58}{x^{19}}+\frac{62}{x^{20}}+\frac{66}{x^{21}}$$

$$\frac{70}{x^{22}}+\frac{72}{x^{23}}+\frac{76}{x^{24}}+\frac{78}{x^{25}}+\frac{78}{x^{26}}+\frac{80}{x^{27}}+\frac{80}{x^{28}}$$

This allows a scientist to visualize quickly some patterns in a series expansion for instance. Here the nth coefficient have value 2∗q(n) where q is the 'restricted partition' function, e.g the number of ways that a number can be represented as the sum of different other numbers.. We can also check the convergence of the infinite product $\prod_{i=0}^{i=n}(1+1/x_n^i)$ to 2, when $x_n \to \infty$:

$$\text{eval}\left(\prod_{i=0}^{200}(1000000000\hat{\ }(-i)+1)\right)$$

We get: 2.000000002

These small examples show how useful a CAS can be. Of course it does not replace the "manual" scientific approach using pen and paper and thinking but it can save a lot of time by performing symbolic computing.

2.4 Matrix operations

The CASIO CAS can compute the determinant of a matrix :

$$\det\left(\begin{bmatrix} 2 & 3 & x-1 & 2 \\ 0 & x & 1 & 1 \\ x & 1 & -2 & x\verb|^|2 \\ -3 & 0 & 4 & -1 \end{bmatrix}\right)$$

The CAS will return:

$$-3\cdot x^4-4\cdot x^3+16\cdot x^2-20\cdot x+19$$

This is obviously a gain of time rather than computing manually the expression:

$2(x(2-4x^2)-(-1-4)+x(3(-1-4)-x(-x+1-8))+3(3(x^2+2)-x(x^2(x-1)+4)+(x-1-2)))$

2.5 Differential equations

The CAS can symbolically solve several classes of differential equations like the linear differential equation of rank one or two.

dSolve(x''−2•x=1, t, x)

The CAS returns the solutions as parameterized functions of t :

$$\left\{x=e^{\sqrt{2}\cdot t}\cdot \mathrm{const}(2)+e^{-\sqrt{2}\cdot t}\cdot \mathrm{const}(1)-\frac{1}{2}\right\}$$

It's not difficult to check that

$x" = C_2(\sqrt{2})^2 exp(\sqrt{2}.t) + C_1(\sqrt{2})^2 exp(-\sqrt{2}.t)$

And that indeed$x" = 2x + 1$.

More complex linear differential equations can also be solved symbolically:

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dSolve(x"-2x+sin(t)^2=0,t,x)
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The CAS returns:

$$\left\{x=e^{\sqrt{2}\cdot t}\cdot \text{const}(2)+e^{-\sqrt{2}\cdot t}\cdot \text{const}(1)-\frac{\cos(2\cdot t)}{12}+\frac{1}{4}\right\}$$

We check that the function $f(t) = cos(2t)/12 + 1/4$is indeed such that:

$f"(t) - 2f(t) = -4cos(2t)/12 - 2cos(2t)/12 - 1/2$

$= -(cos(2t) + 1)/2 = -sin^2(t)$

And the expected solution is the superposition of the solutions of $x" - 2x = 0$with the special solution$f$.

Additionally the CAS can work with Fourier transforms or Laplace Transforms as well as using the FFT algorithm.

There are many other functions provided by the CAS, note that the calculator also offers many non-CAS functions such as graphing, trigonometry etc...

Other calculator models provided with a CAS may offer different functions but roughly they fall in the same range.

The CAS from the CASIO, TI or HP calculators are “basic" CAS - yet as we have seen they are already quite powerful.

3 CAS for PCs

3.1 Introduction

Other classes of CAS - available for PCs - allow more complex symbolic manipulations and computations, in what follows, we are going to detail what they can offer. A good list of

CAS software can be found here.

Several of these CAS are doing approximately the same thing as the CAS of the scientific calculators.

More complex things that some of these CAS can do are the following:

- Computations with finite groups;
- Computations with finite fields, like polynomial fields over a finite field;
- Computations with tensors;
- Modular arithmetic;
- Number theory;
- Algebraic geometry computations;
- Symbolic solvers of complex classes of differential equations.

Here are the main "general purposes" CAS:

Maxima
Magma
MuPAD
SciLab
Matlab
Matematica

Pari is a number theory CAS. Magnus is a group theory oriented CAS. Macaulay2 is focused more on commutative algebra.

There are also CAS for Tensor Theory (Cadabra), combinatorics (GAP) and Algebraic geometry (KANT)

3.2 Magma

Magma is a noncommercial paid CAS developed and maintained by the CAS group at the University of Sydney.

It can be used to perform symbolic computation in the following areas:

Sets, Sequences, Mappings, Basic Rings,Matrices And Linear Algebra, Lattices And Quadratic Forms, Global Arithmetic Fields, Local Arithmetic Fields, Modules, Finite Groups, Finitely-presented Groups, Algebras, Representation Theory, Lie Theory, Commutative Algebra, Algebraic Geometry, Arithmetic Geometry, Modular Arithmetic Geometry, Topology, Geometry, Combinatorics, Coding Theory, Cryptography and Optimization.

Here we see some of its capacities:

We are able to quickly find solutions of diophantine equations using the SET package:

For example, we wish to know the triples of integers (a,b,c) such that:

$1 \leq a \leq 10$
$1 \leq b \leq 10$
$1 \leq c \leq 10$

$a^2 + 2b^2 = c^2$

We input:

{ <a, b, c> : a, b, c in [1..20] | aˆ2 +2∗bˆ2 eq cˆ2 };

Magma returns:

{ <3, 6, 9>, <4, 8, 12>, <7, 6, 11>, <2, 4, 6>, <1, 2, 3>, <1, 12, 17>, <5, 10, 15>, <17, 6, 19>, <6, 12, 18>, <14, 8, 18>, <7, 4, 9> }

Magma can deal with Groups. For instance we can define Abelian groups with finite generators and specify the relations between the generators.
It's just a tiny amount of what magma can do.

Conclusion: CAS for electronic calculators are already quite powerful and possess important features allowing their users to manipulate symbolic equations and symbolic data. Nevertheless CAS for PCs such as Magma largely outperforms them.